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A model of damage coupled to wear

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Abstract

The present paper presents a model of damage coupled to wear. The damage model is based on a continuum model including the gradient of the damage variable. Such a model is non-local in the sense that the evolution of damage is governed by a boundary-value problem instead of a local evolution law. Thereby, the well-known mesh-dependency observed for local damage models is removed. Another feature is that the boundary conditions can be used to introduce couplings between bulk damage and processes at the boundary. In this work such a coupling is suggested between bulk damage and wear at the contact interface. The model is regarded as a first attempt to formulate a continuum damage model for studying crack initiation in fretting fatigue.

The model is given within a thermodynamic framework, where it is assured that the principles of thermodynamics are satisfied. Furthermore, two variational formulations of the full initial boundary value problem, serving as starting points for finite element discretization, are presented. Finally, preliminary numerical results for a simple one-dimensional example are presented and discussed. It is qualitatively shown how the evolution of damage may influence the wear behaviour and how damage may be initiated by the wear process.

Keywords: Continuum thermodynamics; Internal variables; Damage; Gradient of damage; Contact; Friction; Wear; Fretting

1. Introduction

The present paper presents an isotropic damage model coupled to contact, friction and wear. The model is regarded as a first attempt to formulate a continuum model for studying the crack initiation phase in so-called fretting fatigue phenomena, see e.g. Hills and Nowell (1994). Fretting fatigue is observed in a number of machine elements involving mechanical contacts where the contacting bodies are subjected to small relative oscillatory motions, such as splines, shrink-fits, bolted joints, lugs, etc. This type of contact conditions (fretting) decrease the fatigue performance. Typically cracks are developed at or near the contacting surfaces which might grow to a critical size leading to fracture.

The damage model utilized here is much based on the works by Frémond and Nedjar (1996) and Nedjar (2001). The models proposed in these papers include the gradient of the damage variable. Such a model is

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non-local in the sense that the evolution of the damage variable is governed by a boundary value problem instead of a local evolution law. The use of a non-local model removes the well-known mesh-dependency observed when local damage models are solved numerically, see e.g Frémond and Nedjar (1996). Another feature of the non-local model is that the boundary condition of the evolution problem might be used to introduce a coupling between damage and processes at the boundary, for instance chemical reactions. In our case we will assume that wear increases damage. Therefore, a coupling between wear and damage is introduced at the contact interface. Thus, we assume that fretting conditions can be described by an internal state variable measuring wear and that the experimental fact that fatigue performance is decreased when a body is subjected to fretting conditions can be modelled by letting the damage flux depend on this particular variable.

The thermodynamic framework outlined here is somewhat different from that used by Frémond and Nedjar (1996), where the principle of virtual power was modified taking micro-motions of damage into account. Here an internal state variable formalism is used, where a scalar variable and its gradient are internal state variables for measuring damage. In addition two internal state variables are introduced on the contact interface; one measuring tangential slip and another measuring wear. By using these internal state variables together with the displacement and the temperature, a constitutive model is formulated using state laws defined by free energies and complementary laws defined by dual dissipation potentials in the spirit of generalized standard materials (Halphen and Nguyen, 1975), both for the body and the contact interface. The formulation of the state laws for the bulk of the body follows essentially Maugin (1990) and resembles those of other general frameworks for gradient theories, e.g. Svedberg and Runesson (1997) and Lorentz and Andrieux (1999). An important difference between the framework used here and those of the two latter is that all basic principles are in this work given as local postulates, while the formulation of Svedberg and Runesson (1997) and Lorentz and Andrieux (1999) are valid only for global considerations. A way to preserve the local structure of the basic principles is to introduce an extra entropy flux in the second principle of thermodynamics due to Maugin (1990). In fact, it is possible to show that this framework is equivalent to the framework of Frémond and Nedjar (1996). Here the formulation of Maugin (1990) is extended by a coupling to processes at a contact boundary. The way of formulating constitutive laws for a contact interface by a free energy and a dissipation potential, an idea that originates from Frémond (1987, 1988), has been utilized in Klarbring (1990), Johansson and Klarbring (1993) and Strömberg et al. (1996) to formulate constitutive relations for contact, friction and wear.

This study is organized as follows: In Section 2 a general model of damage coupled to wear is derived by use of the principle of virtual power and in accordance with the principles of thermodynamics. In Section 3 a specific model for damage coupled to wear within the proposed thermodynamic framework is suggested and discussed. The model is intended for studying crack initiation in fretting problems. Finally, two variational formulations of the governing equations are given. In Section 4 the behaviour of the suggested model is discussed, based on numerical solutions of a simple one-dimensional example.

2. Derivation of a general model

In this section we derive a general model for isotropic damage of an elastic body in unilateral frictional wearing contact with a rigid support. The model is restricted to quasi-static small-displacement evolutions. Firstly, equilibrium equations and local Clausius–Duhem inequalities are derived from the principle of virtual power and the basic principles of thermodynamics, where an extra entropy flux, as suggested by Maugin (1990), is introduced. Secondly, constitutive assumptions are formulated using state laws defined by gradients (or subdifferentials) of free energies and complementary laws defined by subdifferentials of dual dissipation potentials. The model uses an internal variable representation of the state for the body and

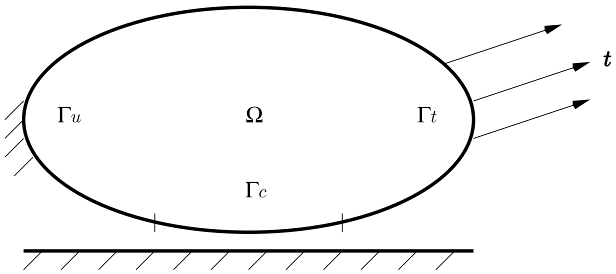

Fig. 1. A deformable body, unilaterally constrained by a rigid obstacle.

the contact interface, respectively. Finally, it is proved that these constitutive assumptions fulfill the Clausius–Duhem inequalities.

Let a region $\Omega \subset \Re^d$ $(d = 2, 3)$ with piecewise smooth boundary $\partial\Omega$ be occupied by a continuously deformable body, see Fig. 1. The boundary $\partial\Omega$ is divided into three disjoint parts: $\Gamma_t \subset \partial\Omega$ where tractions $\boldsymbol{t}$ are prescribed, $\Gamma_u \subset \partial\Omega$ with fixed displacements and $\Gamma_c \subset \partial\Omega$ which is unilaterally constrained by a fixed rigid obstacle. This potential contact surface is considered as a material boundary following the idea of Frémond (1987, 1988), i.e. it is possible to define state laws by a free energy and complementary laws by a dissipation potential on Γ_c.

2.1. The principle of virtual power

In this subsection the method of virtual power in the sense of Germain (1973a,b) and Maugin (1980) is used to derive equilibrium equations and to identify the internal forces with the Cauchy stress tensor and the contact traction vector, respectively.

Assuming quasi-static evolutions the principle of virtual power reads: for any part $\mathscr{D} \subset \Omega$,

$$\widehat{P}_i + \widehat{P}_e = 0 \quad \forall \hat{\boldsymbol{v}} \in \mathscr{V}, \tag{1}$$

where $\mathscr{V}$ is the set of kinematically admissible velocity fields in Ω. The virtual power of the internal and external forces, respectively, are defined by

$$\widehat{P}_i = -\int_{\mathscr{D}} \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\hat{\boldsymbol{v}})\,\mathrm{d}V - \int_{\mathscr{D}\cap\Gamma_c} \boldsymbol{p} \cdot \hat{\boldsymbol{v}}\,\mathrm{d}A \tag{2}$$

and

$$\widehat{P}_e = \int_{\partial\mathscr{D}\setminus\Gamma_c} \boldsymbol{t} \cdot \hat{\boldsymbol{v}}\,\mathrm{d}A, \tag{3}$$

where (:) and (·) represent inner products between tensors and vectors, respectively, $\boldsymbol{\sigma}$ is a symmetric internal stress tensor and $\boldsymbol{p}$ is an internal force vector, respectively. Furthermore,

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \tfrac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}}),$$

where ∇ represents the gradient of a scalar or a vector. In the following $\boldsymbol{u}$ will denote the displacement vector.

By inserting (2) and (3) into (1), applying the divergence theorem and the fact that $\mathscr{D} \subset \Omega$ is arbitrary, the following local equilibrium equations are obtained:

$$\mathrm{div}\boldsymbol{\sigma} = \boldsymbol{0} \text{ in } \mathscr{D}, \tag{4}$$

$$\boldsymbol{\sigma n} = \boldsymbol{t} \text{ on } \partial\mathscr{D} \setminus \Gamma_c, \tag{5}$$

$$\boldsymbol{\sigma n} = -\boldsymbol{p} \text{ on } \partial\mathscr{D} \cap \Gamma_c, \tag{6}$$

where $\boldsymbol{n}$ is the outward unit normal vector to $\partial\mathscr{D}$. The symmetry of $\boldsymbol{\sigma}$ and Eqs. (4) and (5) imply that $\boldsymbol{\sigma}$ can be identified as the Cauchy stress tensor. Furthermore, Eq. (6) implies that $\boldsymbol{p}$ can be identified as the contact traction vector (with reversed sign).

In the following development the contact traction vector as well as the displacement vector on Γ_c will be decomposed into normal components and tangential vectors according to

$$p_n = \boldsymbol{p} \cdot \boldsymbol{n}, \quad \boldsymbol{p}_t = (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{p}, \quad u_n = \boldsymbol{u} \cdot \boldsymbol{n}, \quad \boldsymbol{u}_t = (\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})\boldsymbol{u}, \tag{7}$$

where $\boldsymbol{I}$ represents the identity tensor and $\otimes$ is the tensor product of two vectors.

2.2. The principles of thermodynamics

In this subsection we derive a local Clasius–Duhem inequality from the basic principles of thermodynamics, as they are postulated by Maugin (1990), by introducing the Helmholtz free energies of the body and the contact surface, respectively. The derivation is non-classical in the sense that an extra entropy flux related to damage flux is added.

Assuming that the internal heat production is zero and adopting the quasi-static assumption, the two basic principles of thermodynamics are postulated as: for any part $\mathscr{D} \subset \Omega$

$$\dot{\mathscr{E}} = P_e + \mathscr{Q} \tag{8}$$

and

$$\dot{\mathscr{S}} \geqslant -\int_{\partial\mathscr{D}\setminus\Gamma_c} \frac{\boldsymbol{q} + \boldsymbol{\gamma}}{T} \cdot \boldsymbol{n} \, \mathrm{d}A, \tag{9}$$

where $\mathscr{E}$ is the internal energy, P_e is the power of the external forces defined by (3) evaluated for the real velocity field $\dot{\boldsymbol{u}}$, $\mathscr{Q}$ is the heat supply per time unit, $\mathscr{S}$ is the entropy, $\boldsymbol{q}$ is the heat flux vector and T is the absolute temperature. Furthermore, the quantity $\boldsymbol{\gamma}$ is an extra entropy flux connected with other diffusive mechanisms than heat flux. A superimposed dot denotes a time derivative.

Due to the nature of contact, friction and damage, the state is not necessarily smooth in time. Thus, we need eventually to choose between regarding time-derivatives as left-hand derivatives or right-hand derivatives when rates of the state are expressed. This choice must reflect the nature of constitutive relations and affects the interpretation of the Clausius–Duhem inequality. A strong requirement is that one should be able to prove that the Clausius–Duhem inequality is satisfied at each time instant both for left-hand and right-hand time derivatives, a looser requirement is that it is sufficient to show that it is satisfied for left-hand time derivatives. An argument for the latter can be found in Frémond and Nedjar (1996), [1] where it is stated that constitutive relations are relations between quantities at the present state that should be computed using information about the past evolution, i.e. time derivatives are naturally regarded as left-hand derivatives. However, the stronger requirement that the Clausius–Duhem inequality should be satisfied both for left-hand and right-hand time derivatives can be fulfilled also in the non-smooth case by considering special forms of the free energy. In Strömberg (1997) it was shown that if the free energy is the sum

[1] "We know that the constitutive laws are objective relations, i.e. relations which are computed at time t with the available information given by the past history of the material, i.e. by its past evolution. It results that in the constitutive laws the derivatives with respect to the time are left derivatives.", Frémond and Nedjar (1996).

of a smooth function and an indicator function of a convex set, described by a smooth function, this is indeed the case.

The internal energy, the heat supply per time unit and the entropy are defined by

$$\mathscr{E} = \int_{\mathscr{D}} \rho e \, \mathrm{d}V + \int_{\partial\mathscr{D}\cap\Gamma_c} E \, \mathrm{d}A, \tag{10}$$

$$\mathscr{Q} = -\int_{\partial\mathscr{D}\setminus\Gamma_c} \boldsymbol{q}\cdot\boldsymbol{n} \, \mathrm{d}A \tag{11}$$

and

$$\mathscr{S} = \int_{\mathscr{D}} \rho s \, \mathrm{d}V + \int_{\partial\mathscr{D}\cap\Gamma_c} S \, \mathrm{d}A, \tag{12}$$

where ρ is the mass density, e the specific internal energy, s the specific entropy, E the surfacic density of internal energy on Γ_c and S the surfacic density of entropy on Γ_c.

By inserting (1) and (2), evaluated for the real velocity $\dot{\boldsymbol{u}}$, and (10) and (11) into the first law (8), applying the divergence theorem and using the fact that the part $\mathscr{D}$ is arbitrary, the following local statements of the first law are obtained:

$$\rho\dot{e} = \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) - \mathrm{div}\boldsymbol{q} \quad \text{in } \Omega \tag{13}$$

and

$$\dot{E} = \boldsymbol{p}\cdot\dot{\boldsymbol{u}} + \boldsymbol{q}\cdot\boldsymbol{n} \quad \text{on } \Gamma_c. \tag{14}$$

Next, by inserting (12) into (9), applying the divergence theorem, utilizing

$$\mathrm{div}\left(\frac{\boldsymbol{q}}{T}\right) = \frac{1}{T}\mathrm{div}\boldsymbol{q} - \boldsymbol{q}\cdot\frac{\nabla T}{T^2}$$

and utilizing the same identity for γ/T one obtains the following local statements of the second law:

$$\rho\dot{s}T + \mathrm{div}\boldsymbol{q} + \mathrm{div}\gamma - (\boldsymbol{q}+\gamma)\cdot\frac{\nabla T}{T} \geqslant 0 \quad \text{in } \Omega \tag{15}$$

and

$$\dot{S}T - \boldsymbol{q}\cdot\boldsymbol{n} - \gamma\cdot\boldsymbol{n} \geqslant 0 \quad \text{on } \Gamma_c. \tag{16}$$

Furthermore, by inserting (13) and (14) into (15) and (16), respectively, and introducing the specific and surfacic Helmholtz free energies $\psi = e - sT$ and $\Psi = E - ST$, respectively, one can rewrite these inequalities to obtain the following Clausius–Duhem inequalities:

$$\boldsymbol{\sigma} : \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) - \rho\dot{\psi} - \rho s\dot{T} + \mathrm{div}\gamma - (\boldsymbol{q}+\gamma)\cdot\frac{\nabla T}{T} \geqslant 0 \quad \text{in } \Omega \tag{17}$$

and

$$\boldsymbol{p}\cdot\dot{\boldsymbol{u}} - \dot{\Psi} - S\dot{T} - \gamma\cdot\boldsymbol{n} \geqslant 0 \quad \text{on } \Gamma_c. \tag{18}$$

Note that the above arguments can be reversed. That is, (13), (14), (17) and (18) are equivalent to the basic principles of thermodynamics (8) and (9).

2.3. General constitutive assumptions

In this subsection we formulate some general constitutive assumptions for damage coupled to contact, friction and wear as state laws and complementary laws, defined by means of free energies and dual dissipation potentials, respectively. We prove that these constitutive assumptions are sufficient for satisfaction of the local Clausius–Duhem inequalities (17) and (18).

Before defining the free energies, state laws, etc., we need to define our state variables. In addition to the observable variables $\boldsymbol{u}$ (or $\boldsymbol{\epsilon}(\boldsymbol{u})$) and T, we assume that damage, friction and wear can be measured using four internal state variables. Damage is measured using the scalar variable α, and its gradient $\nabla\alpha$, which is treated as a separate variable when the free energies are defined. The damage variable α takes values in the interval $[0, 1]$, where $\alpha = 0$ corresponds to undamaged material and $\alpha = 1$ corresponds to completely damaged material. Friction is measured by introducing the irreversible tangential displacement, $\boldsymbol{u}_t^i$ as an internal state variable. Finally, wear is measured by using the scalar variable ω, interpreted as an extra gap between the body and the support.

For the bulk material we consider a class of free energies defined by

$$\psi = \psi(\boldsymbol{\epsilon}, T, \alpha, \nabla\alpha) \quad \text{in } \Omega, \tag{19}$$

where ψ is assumed to be a smooth function and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{u})$.

For the contact interface we consider the following class of free energies:

$$\Psi = \Psi(u_n, \boldsymbol{u}_t, \boldsymbol{u}_t^i, \omega; \alpha, T) \quad \text{on } \Gamma_c, \tag{20}$$

where Ψ is convex with respect to $(u_n, \boldsymbol{u}_t, \boldsymbol{u}_t^i, \omega)$ and smooth with respect to α and T.

We define the following state laws for the body:

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi}{\partial\boldsymbol{\epsilon}} \quad \text{in } \Omega, \tag{21}$$

$$s = -\frac{\partial\psi}{\partial T} \quad \text{in } \Omega, \tag{22}$$

$$A = -\rho\frac{\partial\psi}{\partial\alpha} + \operatorname{div}\left(\rho\frac{\partial\psi}{\partial(\nabla\alpha)}\right) \quad \text{in } \Omega \tag{23}$$

and

$$\gamma = \rho\frac{\partial\psi}{\partial(\nabla\alpha)}\dot{\alpha} \quad \text{in } \Omega, \tag{24}$$

where Eq. (23) defines a thermodynamic force A related to $\dot{\alpha}$ and Eq. (24) defines the extra entropy flux.

For the contact interface the following state laws are defined:

$$(p_n, \boldsymbol{p}_t, -\boldsymbol{p}_t^i, -\mathscr{W}) \in \partial\Psi(u_n, \boldsymbol{u}_t, \boldsymbol{u}_t^i, \omega; \alpha, T) \quad \text{on } \Gamma_c, \tag{25}$$

$$\mathscr{A} = -\frac{\partial\Psi}{\partial\alpha} \quad \text{on } \Gamma_c \tag{26}$$

and

$$S = -\frac{\partial\Psi}{\partial T} \quad \text{on } \Gamma_c, \tag{27}$$

where ∂ denotes the partial subdifferential with respect to the arguments before ‘;’. Here $\mathscr{A}$ is a force similar to A, $\boldsymbol{p}^i$ is an internal force related to $\dot{\boldsymbol{u}}_t^i$ and $\mathscr{W}$ is an internal force related to $\dot{\omega}$.

Furthermore, we assume the existence of so-called dual dissipation potentials $\phi(A;\pi)$, parameterized by $\pi = (\epsilon, \alpha)$, and $\Phi(\boldsymbol{p}_t^i, \mathscr{W}, \mathscr{R}; \Pi)$, parameterized by $\Pi = (p_n, \omega, \alpha)$ and where $\mathscr{R}$ is defined by

$$\mathscr{R} = \mathscr{A} - \rho \frac{\partial \psi}{\partial(\nabla\alpha)} \cdot \boldsymbol{n}. \tag{28}$$

Note that $\mathscr{R}$ is the driving force for $\dot{\alpha}$ on Γ_c, see Proposition 1 below. The dissipation potentials are convex functions satisfying the conditions:

$$\phi(0;\pi) = 0, \quad 0 \in \partial\phi(0;\pi), \tag{29}$$

$$\Phi(\mathbf{0}, 0, 0; \Pi) = 0 \quad \text{and} \quad (\mathbf{0}, 0, 0) \in \partial\Phi(\mathbf{0}, 0, 0; \Pi). \tag{30}$$

The following complementary laws are defined for the body:

$$\dot{\alpha} \in \partial\phi(A;\pi) \quad \text{in } \Omega \tag{31}$$

and for the contact interface

$$(\dot{\boldsymbol{u}}_t^i, \dot{\omega}, \dot{\alpha}) \in \partial\Phi(\boldsymbol{p}_t^i, \mathscr{W}, \mathscr{R}; \Pi) \quad \text{on } \Gamma_c. \tag{32}$$

Finally we assume that the following inequality holds:

$$-(\boldsymbol{q} + \boldsymbol{\gamma}) \cdot \frac{\nabla T}{T} \geqslant 0 \quad \text{in } \Omega. \tag{33}$$

This is the generalized thermal dissipation inequality due to Maugin (1990). In that paper it was also suggested a law of Fourier type for the quantity $(\boldsymbol{q} + \boldsymbol{\gamma})$.

Let us now prove that Eqs. (19)–(33) are sufficient conditions for satisfaction of the Clausius–Duhem inequalities (17) and (18) when time derivatives are regarded as left-hand derivatives. This follows from the three propositions presented below. The choice of left-hand time derivatives is crucial when establishing Proposition 3.

Proposition 1. *The complementary laws* (31) *and* (32) *and the conditions* (29) *and* (30) *ensure that the following residual dissipation inequalities are satisfied:*

$$A\dot{\alpha} \geqslant 0 \quad \textit{in } \Omega, \tag{34}$$

$$\boldsymbol{p}_t^i \cdot \dot{\boldsymbol{u}}_t^i + \mathscr{W}\dot{\omega} + \mathscr{R}\dot{\alpha} \geqslant 0 \quad \textit{on } \Gamma_c, \tag{35}$$

Proof. Let us prove that (31) is sufficient for satisfaction of (34). The proof is standard and follows the idea of Moreau (1974). The definition of the subdifferential in (31) and the condition $(29)_1$ imply that

$$0 = \phi(0;\pi) \geqslant \phi(A;\pi) + \dot{\alpha}(0 - A).$$

Taken with $(29)_2$, the inequality above implies that

$$A\dot{\alpha} \geqslant \phi(A;\pi) \geqslant 0.$$

The fact that (32) is sufficient for satisfaction of (35) is shown using (30) in a similar way. This concludes the proof. □

Proposition 2. *The free energy* (19), *the state laws* (21)–(24), *the assumption* (33) *and the residual dissipation inequality* (34) *ensure satisfaction of the Clausius–Duhem inequality* (17).

Proof. By using the chain rule of differentiation on (19) and multiplying by ρ it follows that

$$\rho \frac{\partial \psi}{\partial \epsilon} : \dot{\epsilon} + \rho \frac{\partial \psi}{\partial T} \dot{T} + \rho \frac{\partial \psi}{\partial \alpha} \dot{\alpha} + \rho \frac{\partial \psi}{\partial (\nabla \alpha)} \cdot \nabla \dot{\alpha} - \rho \dot{\psi} = 0.$$

By adding this equality and the inequalities (33) and (34), using

$$\rho \frac{\partial \psi}{\partial (\nabla \alpha)} \cdot \nabla \dot{\alpha} = \operatorname{div}\left(\rho \frac{\partial \psi}{\partial (\nabla \alpha)} \dot{\alpha} \right) - \dot{\alpha} \operatorname{div}\left(\rho \frac{\partial \psi}{\partial (\nabla \alpha)} \right)$$

and inserting (21)–(24) the Clausius–Duhem inequality (17) is recovered. This concludes the proof. □

Proposition 3. *The free energy in* (20), *the state laws* (25)–(27), (28) *and the residual dissipation inequality* (35) *ensure satisfaction of* (18).

Proof. Letting t denote time, the left-hand time derivative of Ψ is by definition

$$\begin{aligned} \dot{\Psi} &= \lim_{\substack{\Delta t \to 0 \\ \Delta t > 0}} \frac{\Psi(t) - \Psi(t - \Delta t)}{\Delta t} \\ &= \lim_{\substack{\Delta t \to 0 \\ \Delta t > 0}} \frac{\Psi(t) - \Psi(u_n(t - \Delta t), \boldsymbol{u}_t(t - \Delta t), \boldsymbol{u}_t^i(t - \Delta t), \omega(t - \Delta t); \alpha(t), T(t))}{\Delta t} + \frac{\partial \Psi}{\partial \alpha} \dot{\alpha} + \frac{\partial \Psi}{\partial T} \dot{T}. \end{aligned}$$

Furthermore, the definition of the subdifferential in (25) implies that

$$\begin{aligned} &\Psi(u_n(t - \Delta t), \boldsymbol{u}_t(t - \Delta t), \boldsymbol{u}_t^i(t - \Delta t), \omega(t - \Delta t); \alpha(t), T(t)) \\ &\quad \geqslant \Psi(t) + p_n(u_n(t - \Delta t) - u_n(t)) + \boldsymbol{p}_t \cdot (\boldsymbol{u}_t(t - \Delta t) - \boldsymbol{u}_t(t)) \\ &\quad - \boldsymbol{p}_t^i \cdot (\boldsymbol{u}_t^i(t - \Delta t) - \boldsymbol{u}_t^i(t)) - \mathscr{W}(\omega(t - \Delta t) - \omega(t)), \end{aligned}$$

where the subdifferential is evaluated at time t. Now, dividing by $\Delta t > 0$ and using the definition of left-hand time derivative above one finds that

$$p_n \dot{u}_n + \boldsymbol{p}_t \cdot \dot{\boldsymbol{u}}_t - \boldsymbol{p}_t^i \cdot \dot{\boldsymbol{u}}_t^i - \mathscr{W} \dot{\omega} - \mathscr{A} \dot{\alpha} - S \dot{T} - \dot{\Psi} \geqslant 0,$$

where (26) and (27) have also been used. Finally, by adding the inequality (35) and using (7), (24) and (28), the inequality (18) is recovered. This concludes the proof of the proposition as well as the statement that (19)–(33) are sufficient conditions for satisfaction of the Clausius–Duhem inequalities (17) and (18). □

At this point it is worth to note that for the special case when the free energy of the contact interface is the sum of an indicator function and a smooth function, it is possible to prove that the last proposition above also holds when time derivatives are regarded as right-hand derivatives, see Strömberg (1997).

3. Specific constitutive assumptions

In this section we propose specific forms of the free energies and the dual dissipation potentials, leading to a model of damage coupled to wear. The model is intended for studying crack initiation in fretting problems. Here we do not consider thermal effects. The constitutive equations defining the model of damage coupled to wear are summarized in the end of the section. Finally, two variational formulations of the full problem are presented.

3.1. Constitutive model for the body

For the bulk material we use a simplified version of the model suggested by Frémond and Nedjar (1996). It is simplified in the sense that it does not distinguish between the response in tension and compression. This feature was introduced in the model suggested by Frémond and Nedjar (1996) by defining the free energy and the dissipation potential in such a fashion that only the positive part of the strain tensor will contribute to the evolution of damage.

The proposed model for rate-independent damage, including the gradient of the damage variable, coupled to linear elasticity, is defined by the following free energy:

$$\psi = \frac{1}{2\rho}[(1-\alpha)(\lambda \mathrm{tr}(\boldsymbol{\epsilon})^2 + 2G\boldsymbol{\epsilon} : \boldsymbol{\epsilon}) + c(\nabla\alpha)^2], \quad \alpha \in [0, 1],$$

where tr represents the trace of a tensor, λ and G are Lame's elasticity coefficients and c is a constitutive constant that measures the influence of the damage of a point on its neighborhood, hence controlling the size of the damaged zone. Furthermore, the dual dissipation potential is taken as

$$\phi(A; \boldsymbol{\epsilon}, \alpha) = I_{\mathscr{B}(\boldsymbol{\epsilon},\alpha)}(A),$$

where $I_K(x)$ denotes the indicator function of a set K, i.e. a function that takes the value 0 if $x \in K$ and $+\infty$ otherwise, and

$$\mathscr{B}(\boldsymbol{\epsilon}, \alpha) = \{A : A - \tfrac{1}{2}\alpha(\lambda \mathrm{tr}(\boldsymbol{\epsilon})^2 + 2G\boldsymbol{\epsilon} : \boldsymbol{\epsilon}) - W \leqslant 0\}.$$

Here $W > 0$ is a constitutive parameter representing the strain energy threshold for initiation of damage.

The state laws (21) and (23) imply for this choice of free energy that

$$\boldsymbol{\sigma} = (1-\alpha)(\lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2G\boldsymbol{\epsilon})$$

and

$$A = \tfrac{1}{2}(\lambda \mathrm{tr}(\boldsymbol{\epsilon})^2 + 2G\boldsymbol{\epsilon} : \boldsymbol{\epsilon}) + c\Delta\alpha, \tag{36}$$

where $\Delta = \mathrm{div}\nabla$ denotes the Laplacian. Moreover, the complementary law in (31) implies for the choice of dissipation potential above that

$$A \in \mathscr{B}(\boldsymbol{\epsilon}, \alpha) : \dot{\alpha}(A' - A) \leqslant 0 \quad \forall A' \in \mathscr{B}(\boldsymbol{\epsilon}, \alpha),$$

or, equivalently,

$$\dot{\alpha} \geqslant 0, \quad h \leqslant 0, \quad \dot{\alpha}h = 0,$$

where

$$h = \tfrac{1}{2}(1-\alpha)(\lambda \mathrm{tr}(\boldsymbol{\epsilon})^2 + 2G\boldsymbol{\epsilon} : \boldsymbol{\epsilon}) + c\Delta\alpha - W.$$

Here, the explicit expression of A given by (36) has also been utilized.

3.2. Constitutive model for the contact interface

In this subsection a simple model for the contact interface introducing a coupling between wear and bulk damage is suggested. The model is designed such that the driving force of damage on the contact surface tends to zero as the damage approaches one. In such a manner situations where the coupled wear-damage problem obviously lacks solution is avoided.

The model is defined by the following free energy for the contact interface:

$$\Psi(u_n, \boldsymbol{u}_t, \boldsymbol{u}_t^i, \omega, \alpha) = I_C(u_n, \omega) + I_D(\boldsymbol{u}_t, \boldsymbol{u}_t^i) + \tfrac{1}{2}(1-\alpha)^2\kappa\omega, \tag{37}$$

where κ is a constitutive parameter which is discussed at the end of this section,

$$C = \{(u_n, \omega) : u_n - \omega - g \leqslant 0\},$$

g is the initial contact gap and

$$D = \{(\boldsymbol{u}_t, \boldsymbol{u}_t^i) : \boldsymbol{u}_t - \boldsymbol{u}_t^i = \boldsymbol{0}\}.$$

In addition, the dual dissipation potential of the contact surface is taken to be

$$\Phi(\boldsymbol{p}_t^i, \mathscr{W}, \mathscr{R}; p_n, \alpha) = I_{\mathscr{F}(p_n,\alpha)}(\boldsymbol{p}_t^i, \mathscr{W}) + I_{\mathscr{G}}(\mathscr{R}),$$

where

$$\mathscr{F}(p_n, \alpha) = \{(\boldsymbol{p}_t, \mathscr{W}) : |\boldsymbol{p}_t| + k\mathscr{W}p_n \leqslant \mu(p_n)_+ + k(p_n - \tfrac{1}{2}(1-\alpha)^2\kappa)p_n\},$$

μ is the friction coefficient, k is the wear coefficient, $(x)_+ = \max(0, x)$ and

$$\mathscr{G} = \{\mathscr{R} : \mathscr{R} = 0\}.$$

The state law in (25) implies for this choice of free energy that

$$p_n \geqslant 0, \quad u_n - \omega - g \leqslant 0, \quad p_n(u_n - \omega - g) = 0, \tag{38}$$

$$\mathscr{W} = p_n - \tfrac{1}{2}(1-\alpha)^2\kappa \tag{39}$$

and $\boldsymbol{p}_t = \boldsymbol{p}_t^i$, with $\boldsymbol{p}_t \in \Re^2$ arbitrary. The conditions in (38) are the Karush–Kuhn–Tucker conditions to the following variational principle:

$$p_n \in \mathscr{K}_n : (u_n - \omega - g)(q_n - p_n) \leqslant 0 \quad \forall q_n \in \mathscr{K}_n,$$

where

$$\mathscr{K}_n = \{p_n : p_n \geqslant 0\}.$$

Furthermore, the state law in (26) generates

$$\mathscr{A} = (1-\alpha)\kappa\omega. \tag{40}$$

The complementary law in (32) implies, for the choice of dissipation potential above, a sliding rule of Coulomb type, i.e.

$$\boldsymbol{p}_t \in \widetilde{\mathscr{F}}(p_n) : \dot{\boldsymbol{u}}_t^i \cdot (\boldsymbol{q}_t - \boldsymbol{p}_t) \leqslant 0 \quad \forall \boldsymbol{q}_t \in \widetilde{\mathscr{F}}(p_n) \tag{41}$$

and a wear law of Archard type, i.e.

$$\dot{\omega} = kp_n|\dot{\boldsymbol{u}}_t^i|,$$

where the set

$$\widetilde{\mathscr{F}}(p_n) = \{\boldsymbol{p}_t : |\boldsymbol{p}_t| \leqslant \mu(p_n)_+\},$$

defining Coulomb's cone, is obtained by inserting the explicit expression of $\mathscr{W}$ given by (39) into the set $\mathscr{F}(p_n)$ and on using that $\boldsymbol{p}_t = \boldsymbol{p}_t^i$. Coulomb's law is expressed in (41) as a principle of maximal dissipation. This is equivalent to the following way of expressing Coulomb's law:

$$|\dot{\boldsymbol{u}}_t|\boldsymbol{p}_t = \dot{\boldsymbol{u}}_t\mu(p_n)_+, \quad |\boldsymbol{p}_t| \leqslant \mu(p_n)_+.$$

Here, $\boldsymbol{u}_t = \boldsymbol{u}_t^i$ has also been used, which is a consequence of the definition of the set D. Furthermore, (32) implies $\dot{\alpha}$ arbitrary and $\mathscr{R} = 0$ by the definition of the set $\mathscr{G}$, which in turn yields, using (28) and (40),

$$\boldsymbol{n} \cdot c\nabla\alpha = (1-\alpha)\kappa\omega.$$

3.3. Summary of constitutive equations

In conclusion, the proposed constitutive model for damage coupled to wear can be summarized as follows:

$$\boldsymbol{\sigma} = (1-\alpha)(\lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2G\boldsymbol{\epsilon}) \quad \text{in } \Omega, \tag{42}$$

$$c\Delta\alpha = A - \tfrac{1}{2}(\lambda \mathrm{tr}(\boldsymbol{\epsilon})^2 + 2G\boldsymbol{\epsilon} : \boldsymbol{\epsilon}) \quad \text{in } \Omega, \tag{43}$$

$$A \in \mathscr{B}(\boldsymbol{\epsilon}, \alpha) : \dot{\alpha}(A' - A) \leqslant 0 \quad \forall A' \in \mathscr{B}(\boldsymbol{\epsilon}, \alpha) \quad \text{in } \Omega, \tag{44}$$

$$\boldsymbol{n} \cdot c\nabla\alpha = 0 \quad \text{on } \partial\Omega \setminus \Gamma_c, \tag{45}$$

$$\boldsymbol{n} \cdot c\nabla\alpha = (1-\alpha)\kappa\omega \quad \text{on } \Gamma_c, \tag{46}$$

$$p_n \in \mathscr{K}_n : (u_n - \omega - g)(q_n - p_n) \leqslant 0 \quad \forall q_n \in \mathscr{K}_n \quad \text{on } \Gamma_c, \tag{47}$$

$$\boldsymbol{p}_t \in \widetilde{\mathscr{F}}(p_n) : \dot{\boldsymbol{u}}_t \cdot (\boldsymbol{q}_t - \boldsymbol{p}_t) \leqslant 0 \quad \forall \boldsymbol{q}_t \in \widetilde{\mathscr{F}}(p_n) \quad \text{on } \Gamma_c, \tag{48}$$

$$\dot{\omega} = kp_n|\dot{\boldsymbol{u}}_t| \quad \text{on } \Gamma_c. \tag{49}$$

The first four Eqs. (42)–(45) are bulk properties similar to the damage models suggested by Frémond and Nedjar (1996) and Nedjar (2001). The last three Eqs. (47)–(49) are tribological laws suggested in Strömberg et al. (1996). The remaining Eq. (46) expresses the experimental fact that the fatigue performance usually is decreased when a body is subjected to fretting conditions, see e.g Hills and Nowell (1994). This fact is here modelled by letting the damage flux depend on the amount of wear, which in turn depends on the amount of oscillatory slip. Thus, κ is a constitutive parameter that governs this dependency.

It is also worth to note that the free energy in (37) is chosen in such a way that $\mathscr{A} \to 0$ as $\alpha \to 1$. Another choice of free energy leading to e.g.

$$\boldsymbol{n} \cdot c\nabla\alpha = \kappa\omega \quad \text{on } \Gamma_c$$

would be troublesome. Since ω is non-decreasing the condition above then requires that $\nabla\alpha$ increases. One might think of situations where the only possibility to obtain this is for α to increase where it is already one and this is impossible since $\alpha \in [0,1]$. Thus, the problem might lack solution. Since in the present case $\boldsymbol{n} \cdot c\nabla\alpha \cdot \boldsymbol{n} \to 0$ as $\alpha \to 1$ at the contact this difficulty is avoided.

3.4. Variational formulations

To conclude this section, we present two variational formulations of the full boundary value problem defined by the equilibrium equations (4)–(6) and the constitutive relations summarized in Eqs. (42)–(49). The full problem reads: Given proper initial conditions and a load history $\boldsymbol{t}(t)$ on a time interval $[0,\tau]$ find $\boldsymbol{u} : [0,\tau] \to \mathscr{V}$, $p_n : [0,\tau] \to \widehat{\mathscr{K}}_n$, $\boldsymbol{p}_t : [0,\tau] \to \widehat{\mathscr{F}}(p_n)$, $\alpha : [0,\tau] \to \widehat{\mathscr{T}}$ and $A : [0,\tau] \to \widehat{\mathscr{B}}$ such that for each time $t \in [0,\tau]$

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}V - \int_{\Gamma_t} \boldsymbol{t} \cdot \boldsymbol{v}\,\mathrm{d}A + \int_{\Gamma_c} \boldsymbol{p} \cdot \boldsymbol{v}\,\mathrm{d}A = 0 \quad \forall \boldsymbol{v} \in \mathscr{V}, \tag{50}$$

$$\int_{\Gamma_c} (q_n - p_n)(u_n - \omega - g)\,\mathrm{d}A \leqslant 0 \quad \forall q_n \in \widetilde{\mathscr{K}}_n, \tag{51}$$

$$\int_{\Gamma_c} (\boldsymbol{q}_t - \boldsymbol{p}_t) \cdot \dot{\boldsymbol{u}}_t\,\mathrm{d}A \leqslant 0 \quad \forall \boldsymbol{q}_t \in \widetilde{\mathscr{F}}(p_n), \tag{52}$$

$$\int_\Omega c\nabla\alpha\cdot\nabla\beta\,\mathrm{d}V+\int_\Omega A\beta\,\mathrm{d}V-\frac{1}{2}\int_\Omega(\lambda\mathrm{tr}(\boldsymbol{\epsilon})^2+2G\boldsymbol{\epsilon}:\boldsymbol{\epsilon})\beta\,\mathrm{d}V-\int_{\Gamma_c}(1-\alpha)\kappa\omega\beta\,\mathrm{d}A=0\quad\forall\beta\in\widehat{\mathscr{T}},\tag{53}$$

$$\int_\Omega\dot{\alpha}(A'-A)\,\mathrm{d}V\leqslant 0\quad\forall A'\in\widehat{\mathscr{B}},\tag{54}$$

where, in addition, (42) is inserted and the evolution of ω is governed by (49). Furthermore, the following notations for function spaces are used:

$$\mathscr{V}=\{\boldsymbol{v}:\boldsymbol{v}(\boldsymbol{x})=\boldsymbol{0},\quad \boldsymbol{x}\in\Gamma_u\},$$

$$\widetilde{\mathscr{K}}_n=\{p_n:p_n(\boldsymbol{x})\geqslant 0,\quad \boldsymbol{x}\in\Gamma_c\},$$

$$\widetilde{\mathscr{F}}(p_n)=\{\boldsymbol{p}_t:|\boldsymbol{p}_t(\boldsymbol{x})|\leqslant\mu(p_n(\boldsymbol{x}))_+,\quad \boldsymbol{x}\in\Gamma_c\},$$

$$\widehat{\mathscr{T}}=\{\beta:0\leqslant\beta(\boldsymbol{x})<1,\quad \boldsymbol{x}\in\Omega\}$$

and

$$\widehat{\mathscr{B}}=\{A:A(\boldsymbol{x})-\tfrac{1}{2}\alpha(\boldsymbol{x})(\lambda\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{u}(\boldsymbol{x}))^2+2G\boldsymbol{\epsilon}(\boldsymbol{u}(\boldsymbol{x}):\boldsymbol{\epsilon}(\boldsymbol{u}(\boldsymbol{x})))-W\leqslant 0,\quad \boldsymbol{x}\in\Omega\}.$$

An alternative formulation is obtained by replacing (53) and (54) by the following variational expressions: Find $\alpha:[0,\tau]\to\widehat{\mathscr{T}}$ and $h:[0,\tau]\to\widehat{\mathscr{H}}$ such that for each time $t\in[0,\tau]$

$$\int_\Omega c\nabla\alpha\cdot\nabla\beta\,\mathrm{d}V+\int_\Omega h\beta\,\mathrm{d}V-\frac{1}{2}\int_\Omega(1-\alpha)(\lambda\mathrm{tr}(\boldsymbol{\epsilon})^2+2G\boldsymbol{\epsilon}:\boldsymbol{\epsilon})\beta\,\mathrm{d}V+\int_\Omega W\beta-\int_{\Gamma_c}(1-\alpha)\kappa\omega\beta\,\mathrm{d}A=0\quad\forall\beta\in\widehat{\mathscr{T}}\tag{55}$$

and

$$\int_\Omega\dot{\alpha}(h'-h)\,\mathrm{d}V\leqslant 0\quad\forall h'\in\hat{\mathscr{H}},\tag{56}$$

where

$$\widehat{\mathscr{H}}=\{h:h(\boldsymbol{x})\leqslant 0,\quad \boldsymbol{x}\in\Omega\}.$$

This formulation is obtained by using the alternative formulation of (43) and (44) given at the end of Section 3.1.

The variational formulations presented above are intended to be used as starting points for a finite element discretization of the problem and in a future study it will be investigated if one formulation is perhaps preferred from a numerical point of view.

4. An example

In this section a one-dimensional example is studied in order to discuss the basic behaviour of the fretting damage model and the influence of the various constitutive parameters. The discussion is based on numerical solutions obtained by a finite element approach.

Consider a homogenous bar of length L, Young's modulus E and cross-sectional area A in unilateral wearing contact with a rigid support moving with a given constant velocity ξ, see Fig. 2.

Under the given circumstances the following one-dimensional problem can be stated from Eqs. (4)–(6) and (42)–(49): Find $u(x)$, $\alpha(x)$ and P such that

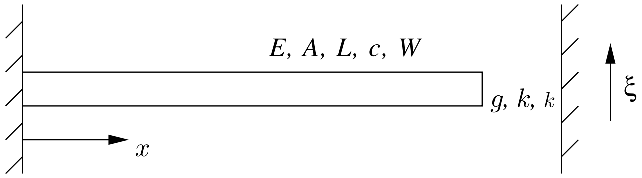

Fig. 2. A bar in unilateral wearing contact with a moving rigid support.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-\alpha)E\frac{\mathrm{d}u}{\mathrm{d}x}\right]=0 \quad \text{in } (0,L),\\
&u(0)=0, \quad (1-\alpha(L))EA\frac{\mathrm{d}u(L)}{\mathrm{d}x}=-P,\\
&P\geqslant 0, \quad u(L)-\omega-g\leqslant 0, \quad P(u(L)-\omega-g)=0,\\
&\dot{\omega}=k\frac{P}{A}\xi,\\
&\dot{\alpha}\geqslant 0, \quad h\leqslant 0, \quad \dot{\alpha}h=0 \quad \text{in } (0,L),\\
&h=\frac{1}{2}(1-\alpha)E\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+c\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}-W \quad \text{in } (0,L)
\end{aligned}
\qquad (57)
$$

and

$$\frac{\mathrm{d}\alpha(0)}{\mathrm{d}x}=0, \quad c\frac{\mathrm{d}\alpha(L)}{\mathrm{d}x}=(1-\alpha(L))\kappa\omega,$$

where P is the normal contact force.

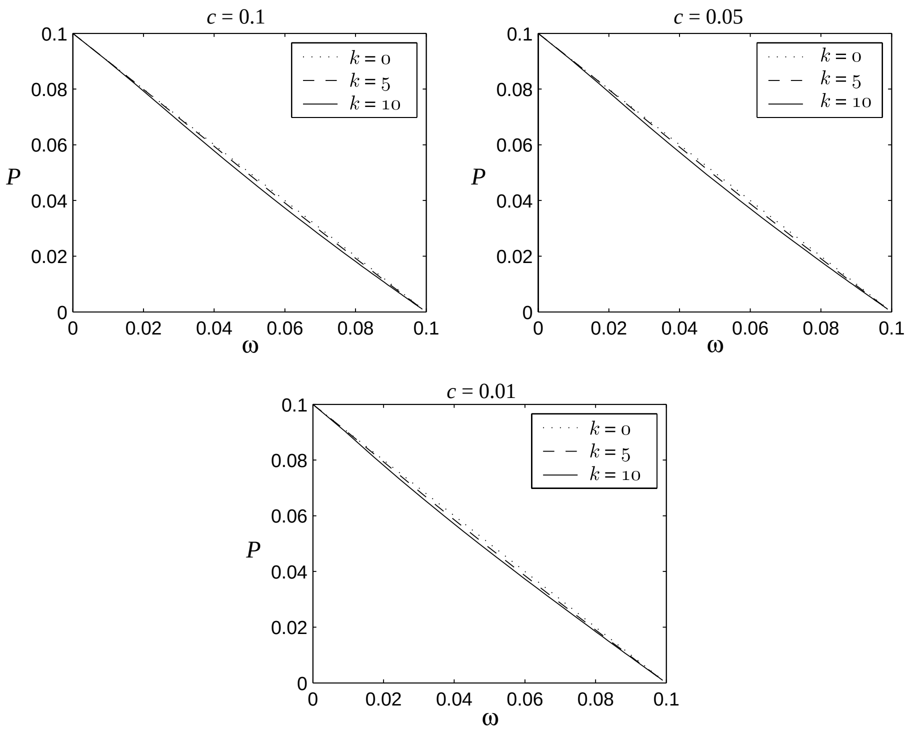

Fig. 3. Contact force versus wear gap evolution for $W = 1$.

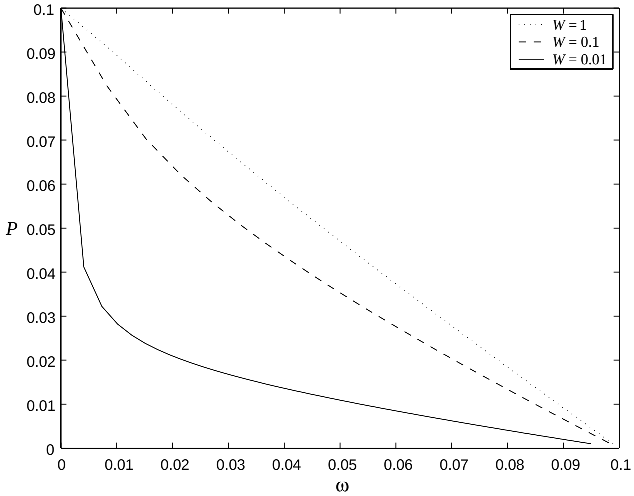

Fig. 4. Contact force versus wear gap evolution for $c = 0.01$, $\kappa = 10$ and different values of W.

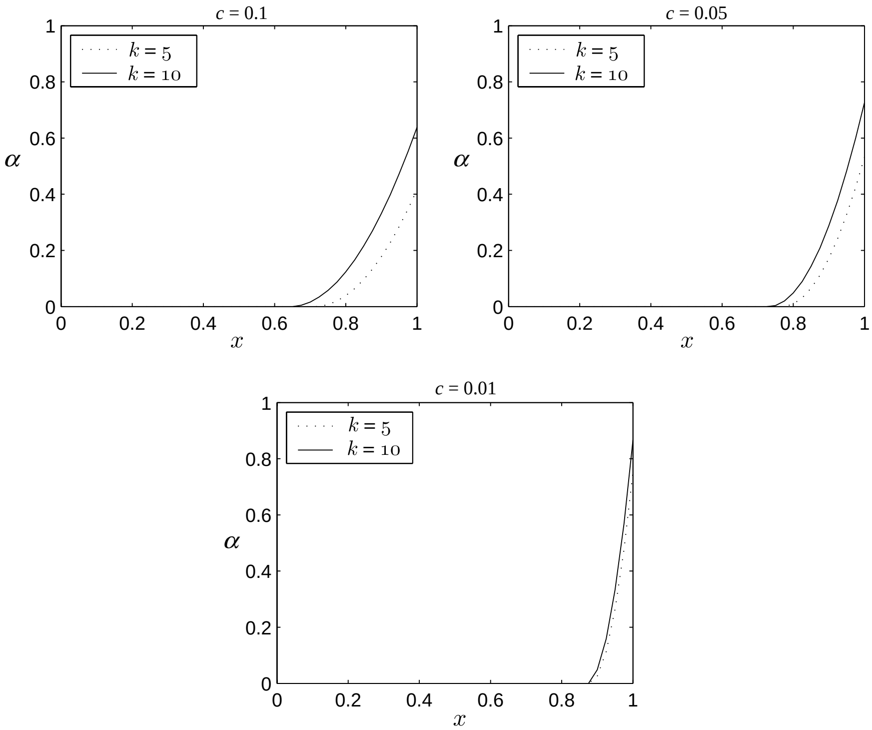

Fig. 5. The distribution of damage when $P = 0$ for $W = 1$.

This problem is solved numerically using a finite element approach based on the one-dimensional correspondence to the second variational formulation presented in the previous section. The fields $u(x)$, $\alpha(x)$

and $h(x)$ are approximated using piecewise linear polynomials and time rates are replaced by Euler-backward differences. Furthermore, the variational inequalities corresponding to the contact conditions and the damage evolution law is restated as equations by means of projections, (Klarbring, 1992). This is done in such a manner that the discrete correspondence to the complementary conditions (57) are satisfied at each nodal point. The result is a system of semi-smooth equations which is solved by a modified Newton method suggested by Pang (1990). The method is similar to the Newton method presented in Ireman et al. (2002). In a forthcoming paper the variational formulations will be studied numerically in detail for two-dimensional problems.

Just to get a qualitative response, we choose $E = 1$ [Pa], $A = 1$ [m^2], $L = 1$ [m], $g = -0.1$ [m], c either equal to 0.1, 0.05 or 0.01 [N], W equal to 1, 0.1 or 0.01 [Pa] and κ equal to 5 or 10 [Pa]. By studying the governing equations it is concluded that k (units [m^2/N]) and ξ (units [m/s]) have no independent influence on the response and their product has no other influence than introducing a time scale: a higher value corresponds to a faster wear process. The calculations are terminated when the contact force vanishes, i.e when the initial gap is worn away. The results are from calculations where 41 nodes were used.

Fig. 3 shows the evolution of the contact force versus the evolution of the wear gap for $W = 1$ and different values of κ and c. For these sets of parameters the contact state is fairly uninfluenced by the damage, i.e. the contact force–wear gap evolution remains close to linear. But as seen from Fig. 4, by lowering the threshold W the contact state becomes more influenced by the evolution of damage.

The distribution of damage when the initial gap is worn away is depicted in Fig. 5 for $W = 1$ and different values of c and κ and in Fig. 6 for $c = 0.01$, $\kappa = 10$ different values of W.

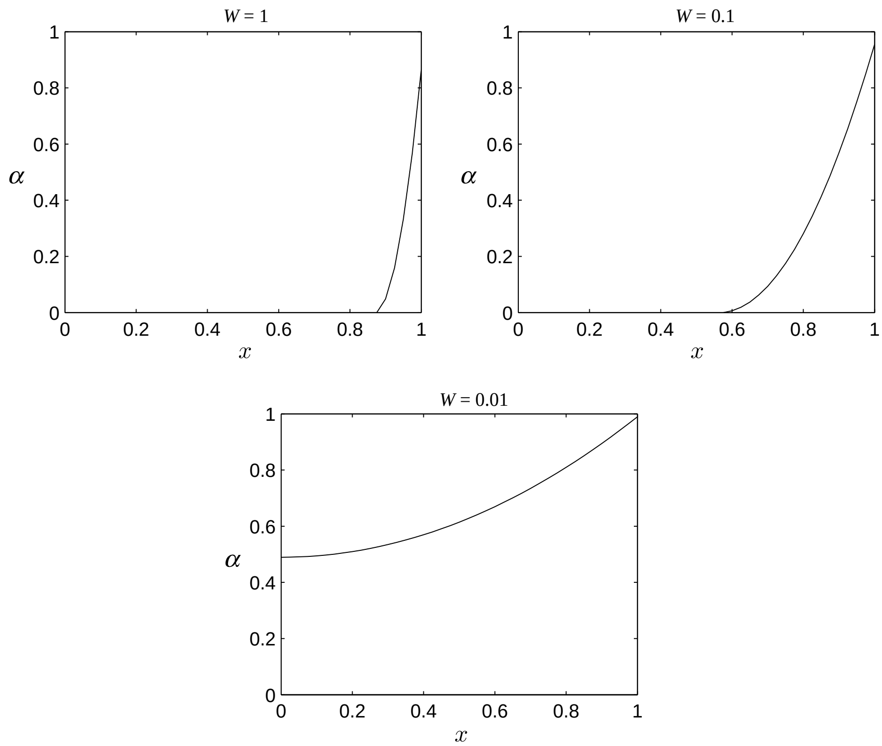

Fig. 6. The distribution of damage when $P = 0$ for $c = 0.01$, $\kappa = 10$ and different values of W.

It is seen from Fig. 5 how a larger value of κ give rise to more damage and how the parameter c controls the size of the damaged zone. From Fig. 6 it is seen that lowering the threshold W also gives rise to more wide-spread damage, e.g. for $W = 0.01$ damage spreads to the entire bar.

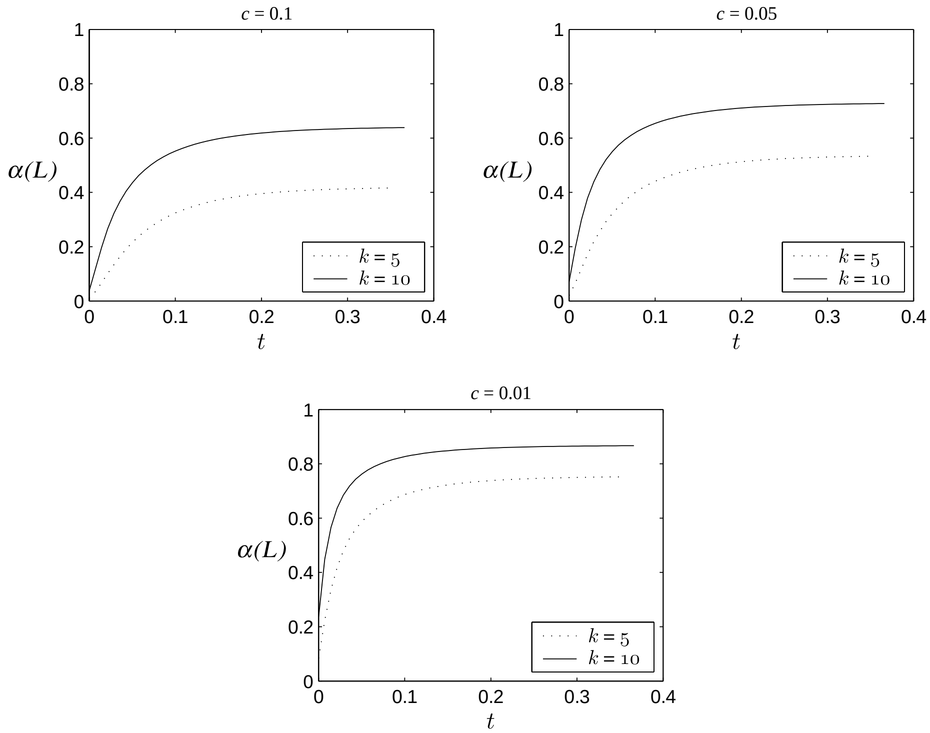

Fig. 7. The evolution of damage at the contact site ($x = L$) for $W = 1$.

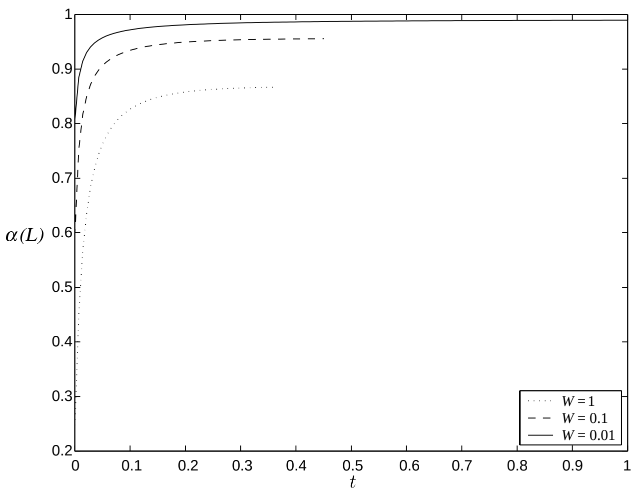

Fig. 8. The evolution of damage at the contact ($x = L$) for $c = 0.01$, $\kappa = 10$ and different values of W.

Finally, let us consider the evolution of the damage at the contact site $x = L$, shown in Figs. 7 and 8. The time scale is chosen such that one time unit is the longest time it takes to wear away the initial gap for the parameters considered (in this case when $c = 0.01$, $W = 0.01$ and $\kappa = 10$).

5. Concluding remarks

The objective of the present work has been to present an as simple model as possible for isotropic damage coupled to wear, regarded as a first attempt to formulate a continuum model for studying crack initiation in fretting fatigue. The model is based on a continuum damage model including the gradient of the damage variable. This implies that the evolution of this variable is governed by a boundary value problem. The boundary conditions are used to introduce a coupling between bulk damage and wear at the contact interface. In this respect an additional constitutive parameter is introduced. This parameter reflects the experimental fact that the nucleation of micro-cracks at the surface depends on the amplitude of relative slip. The model is established within a thermo-mechanical framework where satisfaction of the principles of thermodynamics is assured. A simple one-dimensional example shows the basic behaviour of the model, i.e. quantitatively how the different parameters influence the coupling between damage and wear.

Our future plans is to develop a numerical method for solving the full boundary value problem presented above and to provide numerical results for more representative two-dimensional examples using relevant values of the constitutive parameters. Furthermore, the model needs to be further developed, for instance by including a coupling to plasticity in order to better describe the evolution of damage in a ductile material. Finally, in order to study fatigue crack initiation in metallic components in the high cycle fatigue regime, one should also consider fatigue damage models.

Acknowledgement

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References

Frémond, M., 1987. Adhérence des solides. Journal de Méchanique théorique et appliquée 6, 383–407.

Frémond, M., 1988. Contact with adhesion. In: Moreau, J.J, Panagiotopoulos, P.D., Strang, G. (Eds.), Topics in Nonsmooth Mechanics. Birkäuer Verlag, Basel, pp. 177–221.

Frémond, M., Nedjar, B., 1996. Damage, gradient of damage and principle of virtual power. International Journal of Solids and Structures 33, 1083–1103.

Germain, P., 1973a. La méthode des puissances virtuelles en mécanique des milieux continus. Journal de Méchanique 12, 235–274.

Germain, P., 1973b. The method of virtual power in continuum mechanics Part 2: Microstructure. SIAM Journal of Applied Mathematics 25, 556–575.

Halphen, B., Nguyen, Q.S., 1975. Sur les metériaux standards généralisés. Journal de Mechanique 14, 39–62.

Hills, D.A., Nowell, D., 1994. Mechanics of fretting fatigue, Kluwer Academic Publications, Dordrecht.

Ireman, P., Klarbring, A., Strömberg, N., 2002. Finite element algorithms for thermoelastic wear problems. European Journal of Mechanics/A Solids 21, 423–440.

Johansson, L., Klarbring, A., 1993. Thermoelastic frictional contact problems: modelling, FE-approximation and numerical realization. Computer Methods in Applied Mechanics and Engineering 105, 181–210.

Klarbring, A., 1990. Derivation and analysis of rate boundary-value problems of frictional contact. European Journal of Mechanics, A/Solids 9, 53–85.

Klarbring, A., 1992. Mathematical programming and augmented Lagrangian methods for frictional contact problems. In: Curnier, A. (Ed.), Proceedings of the Contact Mechanics International Symposium, pp. 409–422.

Lorentz, E., Andrieux, S., 1999. A variational formulation for nonlocal damage models. International Journal of Plasticity 15, 119–138.

Maugin, G.A., 1980. The method of virtual power in continuum mechanics: application to coupled fields. Acta Mechanica 35, 1–70.

Maugin, G.A., 1990. Internal variables and dissipative structures. Journal of Non-Equilibrium Thermodynamics 15, 173–192.

Moreau, J.J., 1974. On unilateral constraints, friction and plasticity. In: Capriz, G., Stampacchia, G. (Eds.), New variational techniques in mathematical physics. Edizione cremonese, Rome.

Nedjar, B., 2001. Elastoplastic-damage modelling including the gradient of damage: formulation and computational aspects. International Journal of Solids and Structures 38, 5421–5451.

Pang, J.S., 1990. Newton's method for B-differentiable equations. Mathematics of Operations Research 15, 311–341.

Strömberg, N., Johansson, L., Klarbring, A., 1996. Derivation and analysis of a generalized standard model for contact friction and wear. International Journal of Solids and Structures 33, 1817–1836.

Strömberg, N., 1997. An augmented Lagrangian method for fretting problems. European Journal of Mechanics, A/Solids 16, 573–593.

Svedberg, T., Runesson, K., 1997. A thermodynamically consistent theory of gradient-regularized plasticity coupled to damage. International Journal of Plasticity 13 (6–7), 669–696.